LESSON MAKER PRO

Exact Angle Toolkit

Use angle-sum and angle-difference formulas to find exact values.

HIGH SCHOOL · HSF-TF.C.9

CCSS.Math.Content.HSF-TF.C.9 "(+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems."

NAME _____ DATE _____

COMBINE TWO ANGLES

For any angles a and b, sin(a+b)=sin(a)cos(b)+cos(a)sin(b), sin(a-b)=sin(a)cos(b)-cos(a)sin(b), cos(a+b)=cos(a)cos(b)-sin(a)sin(b), and cos(a-b)=cos(a)cos(b)+sin(a)sin(b). Also, tan(a+b)=(tan(a)+tan(b))/(1-tan(a)tan(b)) and tan(a-b)=(tan(a)-tan(b))/(1+tan(a)tan(b)), when the expressions are defined.

WORKED EXAMPLE

Find cos(105°) exactly.

1. Use 105°=60°+45° and the cosine addition formula.
2. cos(105°)=cos(60°)cos(45°)-sin(60°)sin(45°).
3. cos(105°)=(1/2)(sqrt(2)/2)-(sqrt(3)/2)(sqrt(2)/2).
4. Simplify: cos(105°)=(sqrt(2)-sqrt(6))/4.

Answer: (sqrt(2)-sqrt(6))/4

PRACTICE 6 problems

1 Find sin(75°) exactly.

ANSWER _____

2 Find cos(15°) exactly.

ANSWER _____

3 Find tan(75°) exactly.

ANSWER _____

4 Find tan(15°) exactly.

ANSWER _____

5 Angles x and y are acute. If sin(x)=3/5, cos(x)=4/5, sin(y)=5/13, and cos(y)=12/13, find sin(x+y).

ANSWER _____

6 If tan(x)=1/2 and tan(y)=1/3, find tan(x-y).

ANSWER _____

APPLY IT Show your work

1 At a student music festival, a spotlight is aimed 75° above the level ground. The light reaches a point 6 meters horizontally from the base of the light. Find the exact height of that point above the ground.

WORK SPACE

ANSWER _____

2 A game artist programs a character to move 12 units at an angle of 15° above the positive horizontal axis. Give the exact horizontal component and exact vertical component of the movement.

WORK SPACE

ANSWER _____

CHALLENGE One step up

Let R(t) be the rotation matrix [[cos(t),-sin(t)],[sin(t),cos(t)]]. A rotation through a followed by a rotation through b is the same as a rotation through a+b. Multiply R(b)R(a) and compare its first column with the first column of R(a+b) to prove the sine and cosine addition formulas. Then use tan(t)=sin(t)/cos(t) to derive the tangent addition formula. Finally, explain how replacing b with -b gives all three subtraction formulas.

WORK SPACE AND ANSWER

ANSWER KEY

Exact Angle Toolkit

PART 1 · PRACTICE

#	PROBLEM	ANSWER
1	Find sin(75°) exactly.	**(sqrt(6)+sqrt(2))/4**
2	Find cos(15°) exactly.	**(sqrt(6)+sqrt(2))/4**
3	Find tan(75°) exactly.	**2+sqrt(3)**
4	Find tan(15°) exactly.	**2-sqrt(3)**
5	Angles x and y are acute. If sin(x)=3/5, cos(x)=4/5, sin(y)=5/13, and cos(y)=12/13, find sin(x+y).	**56/65**
6	If tan(x)=1/2 and tan(y)=1/3, find tan(x-y).	**1/7**

PART 2 · APPLY IT

1. **6(2+sqrt(3)) meters** Use height=6tan(75°). Since tan(75°)=2+sqrt(3), the height is 6(2+sqrt(3)) meters.
2. **Horizontal component: 3sqrt(6)+3sqrt(2) units; vertical component: 3sqrt(6)-3sqrt(2) units** Use horizontal=12cos(15°) and vertical=12sin(15°). Substitute the exact angle-difference values for cos(15°) and sin(15°).

CHALLENGE

R(b)R(a) has first column [cos(b)cos(a)-sin(b)sin(a), sin(b)cos(a)+cos(b)sin(a)]. Since R(b)R(a)=R(a+b), cos(a+b)=cos(a)cos(b)-sin(a)sin(b) and sin(a+b)=sin(a)cos(b)+cos(a)sin(b). Therefore tan(a+b)=[sin(a)cos(b)+cos(a)sin(b)]/[cos(a)cos(b)-sin(a)sin(b)]=(tan(a)+tan(b))/(1-tan(a)tan(b)), when defined. Replace b with -b, using cos(-b)=cos(b), sin(-b)=-sin(b), and tan(-b)=-tan(b), to get sin(a-b)=sin(a)cos(b)-cos(a)sin(b), cos(a-b)=cos(a)cos(b)+sin(a)sin(b), and tan(a-b)=(tan(a)-tan(b))/(1+tan(a)tan(b)).

Accept algebraically equivalent radical forms, including rationalized or factored forms. For the challenge, accept a correct matrix-composition argument that compares corresponding entries or the first column.