LESSON MAKER PRO

Choose to Expand

Verify binomial coefficients by counting choices.

HIGH SCHOOL · A3.A.11

A3.A.11 "Verify the Binomial Theorem by mathematical induction or by a combinatorial argument."

NAME _____ DATE _____

COUNT THE CHOICES

To expand a product of identical sums, choose one part from each factor and multiply your choices. If you choose the first part from k of n factors, there are C(n,k) ways to do it. Counting every possible choice verifies the coefficients in the Binomial Theorem.

WORKED EXAMPLE

Verify the expansion of $(x + 3)^3$ by counting choices.

1. To make a term with x^k, choose x from k factors and 3 from the other factors.
2. There are C(3,k) ways to choose those factors. For k = 0, 1, 2, and 3, the terms are 27, 27x, $9x^2$, and x^3.
3. These terms account for every choice from the three factors, so their sum verifies the expansion.

Answer: $(x + 3)^3 = x^3 + 9x^2 + 27x + 27$.

PRACTICE 6 problems

1 In $(x + 1)^4$, how many ways can you choose x from exactly two factors? Use that count to verify the coefficient of x^2.

ANSWER _____

2 In $(x + 2)^5$, choose x from exactly two factors. How many choices are there, and what coefficient of x^2 do they produce?

ANSWER _____

3 Use a choice count to verify the coefficient of x^2 in $(2x + 1)^6$. State the number of choices and the coefficient.

ANSWER _____

4 To make x^6 in $(x - 1)^7$, choose -1 from one factor. How many choices are there, and what coefficient do they produce?

ANSWER _____

5 To verify the coefficient of x^2 in $(x + 1)^8$, split the choices into those that select x from the first factor and those that do not. How many choices are in each group, and what is their total?

ANSWER _____

6 In $(x + 2)^{10}$, choose 2 from exactly one factor and x from all the others. How many choices are there, and what is the resulting term?

ANSWER _____

APPLY IT Show your work

1 A photo app has 5 filters, and each filter can be on or off. Let x stand for an on filter and 1 for an off filter. Explain why the coefficient of x^2 in $(1 + x)^5$ counts settings with exactly two filters on. Find that coefficient.

WORK SPACE

ANSWER _____

2 You are making a beat with 6 slots. Each slot can be silent or contain one of two different clips. In $(1 + 2x)^6$, x marks a slot that contains a clip. Explain why the coefficient of x^2 counts beats with clips in exactly two slots, and find the coefficient.

WORK SPACE

ANSWER _____

CHALLENGE One step up

For any nonnegative integer n and any integer k from 0 through n, explain why the coefficient of x^k in $(x + 2)^n$ is $C(n,k)2^{n-k}$. Explain why this verifies the full binomial expansion, not just one coefficient.

WORK SPACE AND ANSWER

Choose to Expand

PART 1 · PRACTICE

#	PROBLEM	ANSWER
1	In $(x + 1)^4$, how many ways can you choose x from exactly two factors? Use that count to verify the coefficient of x^2.	**C(4,2) = 6 ways; the coefficient is 6.**
2	In $(x + 2)^5$, choose x from exactly two factors. How many choices are there, and what coefficient of x^2 do they produce?	**C(5,2) = 10 choices; the coefficient is $10 \cdot 2^3 = 80$.**
3	Use a choice count to verify the coefficient of x^2 in $(2x + 1)^6$. State the number of choices and the coefficient.	**C(6,2) = 15 choices; the coefficient is $15 \cdot 2^2 = 60$.**
4	To make x^6 in $(x - 1)^7$, choose -1 from one factor. How many choices are there, and what coefficient do they produce?	**There are 7 choices; the coefficient is -7.**
5	To verify the coefficient of x^2 in $(x + 1)^8$, split the choices into those that select x from the first factor and those that do not. How many choices are in each group, and what is their total?	**C(7,1) = 7 choices select x from the first factor; C(7,2) = 21 do not. The total is 28, so the coefficient is 28.**
6	In $(x + 2)^{10}$, choose 2 from exactly one factor and x from all the others. How many choices are there, and what is the resulting term?	**There are 10 choices; the term is $20x^9$.**

PART 2 · APPLY IT

1. **The coefficient is C(5,2) = 10, so there are 10 settings with exactly two filters on.** Each factor represents one filter. Choosing x from exactly two of the five factors selects the two filters that are on.
2. **The coefficient is $C(6,2) \cdot 2^2 = 60$, so there are 60 such beats.** Choose the two slots that contain clips in C(6,2) = 15 ways. Each selected slot has two clip choices, giving $15 \cdot 4 = 60$ beats.

CHALLENGE

To make x^k, choose x from k of the n factors and 2 from the rest. There are C(n,k) such choices, each producing $2^{n-k}x^k$. Every product in the expansion comes from one of these choices for some k from 0 through n, so the formula accounts for every coefficient.

Accept equivalent choice-count notation and equivalent expanded terms. In the challenge, students must explain both the coefficient for a given k and why the cases cover every product.