LESSON MAKER PRO

Proofs That Grow

Use one case to prove the next.

HIGH SCHOOL · A3.S.46

A3.S.46 "Prove statements using mathematical induction."

NAME _____ DATE _____

THE INDUCTION PATTERN

To prove a statement for every whole number $n \geq 1$, first check $n = 1$. Then assume it is true for $n = k$ and use that assumption to prove it for $n = k + 1$.

WORKED EXAMPLE

Prove that $11 + 22 + \ldots + 11n = 11n(n + 1)/2$ for every $n \geq 1$.

1. Base case: When $n = 1$, both sides equal 11.
2. Assume $11 + 22 + \ldots + 11k = 11k(k + 1)/2$.
3. Add the next term: $11k(k + 1)/2 + 11(k + 1) = 11(k + 1)(k + 2)/2$.

Answer: The base case holds, and the assumed case leads to the next case. The formula is true for every $n \geq 1$.

PRACTICE 6 problems

1 Prove $2 + 4 + \ldots + 2n = n(n + 1)$ for $n \geq 1$. Show the base case, then assume the formula for k and simplify the sum after adding $2(k + 1)$.

ANSWER _____

2 Prove $1 + 3 + \ldots + (2n - 1) = n^2$ for $n \geq 1$. Show the base case, then assume the formula for k and add the next odd number.

ANSWER _____

3 Prove $1 + 2 + \ldots + n = n(n + 1)/2$ for $n \geq 1$. Show the base case, then assume the formula for k and add $k + 1$.

ANSWER _____

4 Prove $2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1$ for $n \geq 1$. Show the base case, then assume the formula for k and add 2^k.

ANSWER _____

5 Prove $1/3 + 1/9 + \ldots + 1/3^n = (1 - 1/3^n)/2$ for $n \geq 1$. Show the base case, then assume the formula for k and add $1/3^{k+1}$.

ANSWER _____

6 Prove $1/(1\cdot2) + 1/(2\cdot3) + \ldots + 1/(n(n + 1)) = n/(n + 1)$ for $n \geq 1$. Show the base case, then assume the formula for k and add $1/((k + 1)(k + 2))$.

ANSWER _____

APPLY IT Show your work

1 In a game app, each try has an independent 1/4 chance of winning a prize. Let P(n) be the probability of winning at least once in n tries. Prove by induction that $P(n) = 1 - (3/4)^n$ for $n \geq 1$. For the induction step, split the outcomes into those with an earlier win and those with a first win on the next try.

WORK SPACE

ANSWER _____

2 A coin-flip game pays a bonus when exactly one of n fair flips is heads. Let Q(n) be the probability of earning the bonus. Prove by induction that $Q(n) = n/2^n$ for $n \geq 1$. For the induction step, consider whether the last flip is heads or tails.

WORK SPACE

ANSWER _____

CHALLENGE One step up

Prove by induction that $1/2 + 2/2^2 + \ldots + n/2^n = 2 - (n + 2)/2^n$ for $n \geq 1$. Show the base case and simplify the step after adding $(k + 1)/2^{k+1}$.

WORK SPACE AND ANSWER

Proofs That Grow

PART 1 · PRACTICE

#	PROBLEM	ANSWER
1	Prove $2 + 4 + \ldots + 2n = n(n + 1)$ for $n \geq 1$. Show the base case, then assume the formula for k and simplify the sum after adding $2(k + 1)$.	**Base: $2 = 1(2)$. Step: $k(k + 1) + 2(k + 1) = (k + 1)(k + 2)$.**
2	Prove $1 + 3 + \ldots + (2n - 1) = n^2$ for $n \geq 1$. Show the base case, then assume the formula for k and add the next odd number.	**Base: $1 = 1^2$. Step: $k^2 + (2k + 1) = (k + 1)^2$.**
3	Prove $1 + 2 + \ldots + n = n(n + 1)/2$ for $n \geq 1$. Show the base case, then assume the formula for k and add $k + 1$.	**Base: $1 = 1(2)/2$. Step: $k(k + 1)/2 + (k + 1) = (k + 1)(k + 2)/2$.**
4	Prove $2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1$ for $n \geq 1$. Show the base case, then assume the formula for k and add 2^k.	**Base: $2^0 = 1 = 2^1 - 1$. Step: $(2^k - 1) + 2^k = 2^{k+1} - 1$.**
5	Prove $1/3 + 1/9 + \ldots + 1/3^n = (1 - 1/3^n)/2$ for $n \geq 1$. Show the base case, then assume the formula for k and add $1/3^{k+1}$.	**Base: $1/3 = (1 - 1/3)/2$. Step: $(1 - 1/3^k)/2 + 1/3^{k+1} = (1 - 1/3^{k+1})/2$.**
6	Prove $1/(1\cdot2) + 1/(2\cdot3) + \ldots + 1/(n(n + 1)) = n/(n + 1)$ for $n \geq 1$. Show the base case, then assume the formula for k and add $1/((k + 1)(k + 2))$.	**Base: $1/(1\cdot2) = 1/2$. Step: $k/(k + 1) + 1/((k + 1)(k + 2)) = (k + 1)/(k + 2)$.**

PART 2 · APPLY IT

1. **Base: $P(1) = 1/4 = 1 - 3/4$. If $P(k) = 1 - (3/4)^k$, then $P(k + 1) = P(k) + (1 - P(k))/4 = 1 - (3/4)^{k+1}$.** The chance of no win in the first k tries is $1 - P(k)$. One-fourth of those outcomes produce a first win on the next try.
2. **Base: $Q(1) = 1/2$. If $Q(k) = k/2^k$, then $Q(k + 1) = Q(k)/2 + (1/2^k)(1/2) = (k + 1)/2^{k+1}$.** Exactly one head occurs either when the first k flips contain one head and the last flip is tails, or when the first k flips are all tails and the last flip is heads. These cases do not overlap.

CHALLENGE

Base: $1/2 = 2 - 3/2$. Assume the formula holds for k. Then $2 - (k + 2)/2^k + (k + 1)/2^{k+1} = 2 - (k + 3)/2^{k+1}$, which is the formula for $k + 1$.

Accept equivalent algebra in each induction step. A complete proof must check $n = 1$ and state that the step assumes the formula for k.