LESSON MAKER PRO

Induction in Action

Prove a pattern for every positive integer.

HIGH SCHOOL · C.N.3

C.N.3 "Prove statements using mathematical induction."

NAME _____ DATE _____

TWO STEPS MAKE A PROOF

To prove a statement for every integer $n \geq 1$, first show it works when $n = 1$. Then assume it works for $n = k$ and use that assumption to show it works for $n = k + 1$. These steps together prove the statement for all positive integers.

WORKED EXAMPLE

Prove that $4 + 8 + \ldots + 4n = 2n(n + 1)$ for every integer $n \geq 1$.

1. Base case: When $n = 1$, the left side is 4 and the right side is $2(1)(2) = 4$.
2. Assume $4 + 8 + \ldots + 4k = 2k(k + 1)$ for some integer $k \geq 1$.
3. Add the next term: $2k(k + 1) + 4(k + 1) = 2(k + 1)(k + 2)$. This is the required formula for $n = k + 1$.

Answer: The base case and induction step prove the formula for every integer $n \geq 1$.

PRACTICE 6 problems

1 Prove by induction that $5 + 10 + \ldots + 5n = 5n(n + 1)/2$ for every integer $n \geq 1$.

ANSWER _____

2 Prove by induction that $1 + 3 + \ldots + (2n - 1) = n^2$ for every integer $n \geq 1$.

ANSWER _____

3 Prove by induction that $1 + 3 + \ldots + 3^{n-1} = (3^n - 1)/2$ for every integer $n \geq 1$.

ANSWER _____

4 Prove by induction that $7^n - 1$ is divisible by 6 for every integer $n \geq 1$.

ANSWER _____

5 Prove by induction that $2^n \geq n + 1$ for every integer $n \geq 1$.

ANSWER _____

6 Prove by induction that $6 + 13 + 20 + \ldots + (7n - 1) = n(7n + 5)/2$ for every integer $n \geq 1$.

ANSWER _____

APPLY IT Show your work

1 A student practices guitar for 9 minutes on day 1, 18 minutes on day 2, and 9n minutes on day n. Prove by induction that the total after n days is $9n(n + 1)/2$ minutes. Then find the total after 6 days.

WORK SPACE

ANSWER _____

2 In a game, you earn 10 XP at level 1. The XP earned at each new level is twice the XP earned at the level before it. Prove by induction that the total earned through level n is $10(2^n - 1)$ XP. Then find the total through level 5.

WORK SPACE

ANSWER _____

CHALLENGE One step up

Prove by induction that $1^2 + 2^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6$ for every integer $n \geq 1$. Show how you factor the expression in the induction step.

WORK SPACE AND ANSWER

ANSWER KEY

Induction in Action

PART 1 · PRACTICE

#	PROBLEM	ANSWER
1	Prove by induction that $5 + 10 + \ldots + 5n = 5n(n + 1)/2$ for every integer $n \geq 1$.	**Base: $5 = 5(1)(2)/2$. Assume the sum through $5k$ is $5k(k + 1)/2$. Adding $5(k + 1)$ gives $5k(k + 1)/2 + 5(k + 1) = 5(k + 1)(k + 2)/2$, the formula for $n = k + 1$.**
2	Prove by induction that $1 + 3 + \ldots + (2n - 1) = n^2$ for every integer $n \geq 1$.	**Base: $1 = 1^2$. Assume the sum through $2k - 1$ is k^2. The next term is $2k + 1$, so the new sum is $k^2 + 2k + 1 = (k + 1)^2$.**
3	Prove by induction that $1 + 3 + \ldots + 3^{n-1} = (3^n - 1)/2$ for every integer $n \geq 1$.	**Base: $1 = (3^1 - 1)/2$. Assume the sum through 3^{k-1} is $(3^k - 1)/2$. Adding 3^k gives $(3^k - 1)/2 + 3^k = (3^{k+1} - 1)/2$.**
4	Prove by induction that $7^n - 1$ is divisible by 6 for every integer $n \geq 1$.	**Base: $7^1 - 1 = 6$, which is divisible by 6. Assume $7^k - 1$ is divisible by 6. Then $7^{k+1} - 1 = 7(7^k - 1) + 6$, which is also divisible by 6.**
5	Prove by induction that $2^n \geq n + 1$ for every integer $n \geq 1$.	**Base: $2^1 = 2 \geq 1 + 1$. Assume $2^k \geq k + 1$. Then $2^{k+1} \geq 2(k + 1) \geq k + 2$ because $k \geq 1$.**
6	Prove by induction that $6 + 13 + 20 + \ldots + (7n - 1) = n(7n + 5)/2$ for every integer $n \geq 1$.	**Base: $6 = 1(7 + 5)/2$. Assume the sum through $7k - 1$ is $k(7k + 5)/2$. Adding $7(k + 1) - 1 = 7k + 6$ gives $k(7k + 5)/2 + 7k + 6 = (k + 1)(7k + 12)/2$, the formula for $n = k + 1$.**

PART 2 · APPLY IT

1. **The formula holds for every integer $n \geq 1$. After 6 days, the student has practiced for 189 minutes.** At $n = 1$, both sides equal 9. If the total after k days is $9k(k + 1)/2$, adding $9(k + 1)$ gives $9(k + 1)(k + 2)/2$; substituting $n = 6$ gives 189.
2. **The formula holds for every integer $n \geq 1$. The total through level 5 is 310 XP.** At $n = 1$, the total is $10 = 10(2^1 - 1)$. If the total through level k is $10(2^k - 1)$, adding the next level's $10(2^k)$ XP gives $10(2^{k+1} - 1)$; at $n = 5$, this is 310.

CHALLENGE

Base: $1^2 = 1(2)(3)/6$. Assume the sum through k^2 is $k(k + 1)(2k + 1)/6$. Adding $(k + 1)^2$ gives $(k + 1)[k(2k + 1) + 6(k + 1)]/6 = (k + 1)(2k^2 + 7k + 6)/6 = (k + 1)(k + 2)(2k + 3)/6$, the formula for $n = k + 1$.

Accept equivalent algebra in an induction step. Each proof must establish the base case and use the assumption at k to prove the statement at $k + 1$.