

Why Every Circle Is Similar

Build a transformation proof that works for any two circles.

HIGH SCHOOL • HSG-C.A.1

CCSS.Math.Content.HSG-C.A.1 "Prove that all circles are similar."

NAME _____

DATE _____

MAP THE ENTIRE CIRCLE

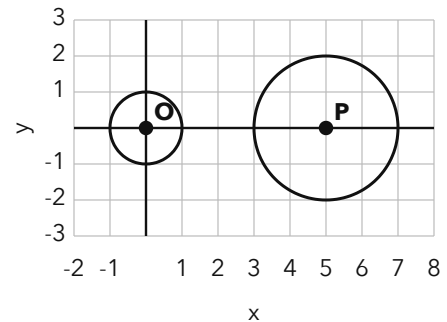
A circle is the set of all points at a fixed positive distance from its center. A translation preserves distances. A dilation with factor $k > 0$ multiplies all distances from its center by k . To prove two circles similar, give a composition of these transformations that maps the entire first circle onto the second.

WORKED EXAMPLE

Circle O has center $(0,0)$, radius 1. Circle P has center $(5,0)$, radius 2. Describe a similarity mapping O's circle onto P's circle.

1. Translate 5 units right. The image now has center P and radius 1.
2. Dilate about P with factor 2. Every radius becomes 2; reversing these transformations recovers the original circle.

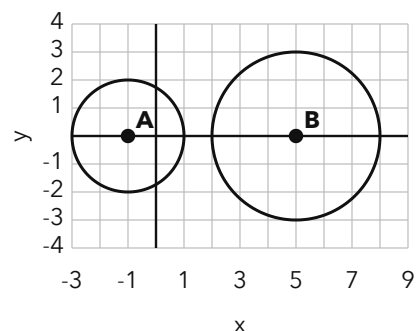
Answer: Translate by $(5,0)$, then dilate about P by 2.



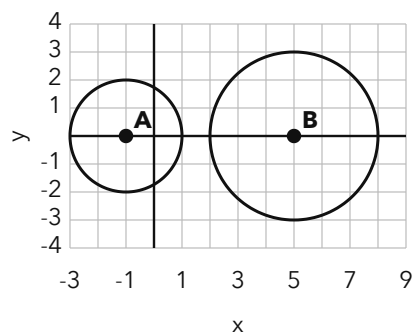
PRACTICE 4 problems

- 1** Circle A has center $(-1,0)$, radius 2; circle B has center $(5,0)$, radius 3. Give a translation followed by a dilation that maps A's circle onto B's circle. Name the dilation center.

ANSWER _____



- 2 Use the same circles shown in Practice 1. A student dilates about A with factor $\frac{3}{2}$ and stops. Explain what is correct, what is missing, and how to finish the mapping.



ANSWER _____

- 3 Now let any source circle have center O and radius $r > 0$, and any target circle have center P and radius $s > 0$. Translate O to P. For an arbitrary source point X, how far is its image X_1 from P? Justify.

ANSWER _____

- 4 Continue Practice 3. Choose a dilation factor k about P so that X_1 maps to a point X_2 on the target circle. Prove the distance condition for every source point X.

ANSWER _____

APPLY IT Show your work

- 1** A designer says the preceding argument proves every target point is reached. Supply the missing reverse argument: start with any target point Y , undo the dilation, then undo the translation. Where does the point end up, and why?

ANSWER _____

- 2** Write a complete proof that ALL circles are similar. Use arbitrary centers O , P and positive radii r , s . Identify the transformations, show where every source point goes, and explain why every target point is reached.

ANSWER _____

CHALLENGE One step up

Why is $r > 0$ needed in your proof? If someone calls the single point O a 'circle of radius 0,' can a translation and a finite positive dilation map it onto a circle of positive radius? Explain.

PRACTICE 1

Translate by $(6,0)$; then dilate about $B(5,0)$ with factor $3/2$.

PRACTICE 2

The radius becomes 3, but the center stays A. Translate by $(6,0)$ to move the center to B. Dilation then translation also works.

PRACTICE 3

$PX_1 = r$: translation preserves the distance $OX = r$.

PRACTICE 4

$k = s/r > 0$. $PX_2 = (s/r)PX_1 = (s/r)r = s$. Since X was arbitrary, every source point maps to the target circle.

APPLY IT 1

Every target point has a preimage on the source circle.

Dilate Y about P by r/s : its distance from P becomes $(r/s)s = r$. Undo the translation, sending P to O .

Distance is preserved, so the resulting point is distance r from O . Reapplying the original map reaches Y .

APPLY IT 2

A translation followed by a positive dilation maps any source circle onto any target circle; therefore all circles are similar.

Translate O to P and dilate about P by s/r . Each source distance r becomes s , so every image lies on the target. Conversely, inverse dilation r/s and inverse translation take each target point to distance r from O . Thus the map is onto. This composition is a similarity.

CHALLENGE

s/r is undefined at $r = 0$. A translation and dilation send a single point to one point, not to an entire positive-radius circle. The positive-radius condition is essential.

This is a general proof, not just a numerical scale-factor exercise. Accept dilation about O followed by translation O to P if both the forward and reverse distance arguments are correct. r and s denote radii, not diameters. The grids illustrate numerical cases; the proof applies to centers anywhere in the plane.