

Arc and Sector Power

Derive radian, arc-length, and sector-area relationships.

HIGH SCHOOL • HSG-C.B.5

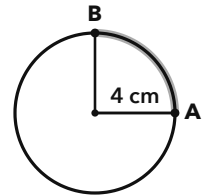
CCSS.Math.Content.HSG-C.B.5 "Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector."

NAME _____

DATE _____

A FIXED ANGLE GIVES A FIXED RATIO

Dilations preserve central angles and scale radii and arcs equally. Radian measure is arc length/radius. A full turn is 2π radians.



WORKED EXAMPLE

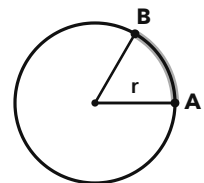
Arc AB is a quarter-circle (90°), with center O and radius 4 cm. Find its length and radian measure.

1. A quarter of circumference $2\pi(4)$ is $s = 2\pi$ cm.
2. $\theta = s/r = 2\pi/4 = \pi/2$ radians; similarity preserves this ratio.

Answer: $s = 2\pi$ cm; $\theta = \pi/2$ radians.

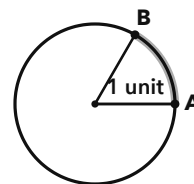
PRACTICE 4 problems

- 1** A sector has radius r and arc length s . A dilation by $k > 0$ creates a similar sector with radius $R = kr$ and arc length $S = ks$. Simplify S/R and explain why a fixed central angle has arc length proportional to radius.



ANSWER _____

- 2** Define θ as the arc length on a UNIT circle cut off by a given central angle. Use Practice 1 to show that $s/r = \theta$ on any circle. Then derive $s = r\theta$ and explain why θ must be in radians.



ANSWER

- 3** A sector has angle θ radians, with $0 < \theta \leq 2\pi$. What fraction of a full turn is it? Use the full circle's area πr^2 to DERIVE the sector-area formula. Show the cancellation.

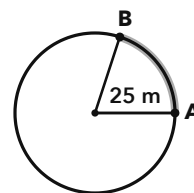
ANSWER

- 4** A sector has radius 9 cm and arc length 9π cm. Find its angle in radians, fraction of a full turn, and area. Show your reasoning.

ANSWER _____

APPLY IT Show your work

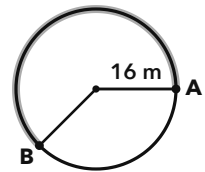
- 1** A circular festival light has radius 25 m. Its lit sector has central angle 72° (the smaller sector between OA and OB). O is the center. Convert the angle to radians, then find the lit arc length and sector area.



ANSWER _____

2

A circular game zone has radius 16 m. Its MAJOR arc AB is 20π m long and borders the larger sector. Find that sector's reflex angle in radians and its area. Explain why you must use the larger sector.



ANSWER _____

CHALLENGE One step up

Using $s = r\theta$ and your area formula, derive $A = (1/2)rs$. For a fixed angle, explain how arc length and sector area change when the radius doubles.

PRACTICE 1

$S/R = ks/(kr) = s/r$. The ratio stays constant for similar sectors with the same angle, so arc length is a constant times radius.

PRACTICE 2

On the unit circle $s/r = \theta/1 = \theta$. Similarity gives the same ratio on any circle, so $s/r = \theta$ and $s = r\theta$. Radian measure is this ratio; degrees use a different numerical scale.

PRACTICE 3

Fraction $\theta/(2\pi)$. $A = [\theta/(2\pi)]\pi r^2 = (1/2)r^2\theta$.

PRACTICE 4

$\theta = s/r = \pi$; fraction $\pi/(2\pi) = 1/2$; $A = (1/2)\pi(9^2) = 81\pi/2 \text{ cm}^2$.

APPLY IT 1

$\theta = 2\pi/5$ radians; arc $10\pi \text{ m}$; area $125\pi \text{ m}^2$.

72° is $1/5$ of a full turn, so $\theta = 2\pi/5$. $s = 25(2\pi/5) = 10\pi$; $A = (1/2)(25^2)(2\pi/5) = 125\pi$.

APPLY IT 2

$\theta = 5\pi/4$ radians (225°); area $160\pi \text{ m}^2$.

$\theta = 20\pi/16 = 5\pi/4$, greater than π . $A = (1/2)(16^2)(5\pi/4) = 160\pi$. The given arc exceeds a semicircle, so the sector is the reflex one.

CHALLENGE

Substitute $\theta = s/r$: $A = (1/2)r^2(s/r) = (1/2)rs$. Doubling r doubles s , so area becomes $(1/2)(2r)(2s) = 4A$. Arc length doubles; area quadruples.

Require the similarity ratio argument and area derivation, not merely recalled formulas. Arc length has linear units and area square units. All angle givens are stated in the task text; students should not measure the schematic circles. In Apply It 2 the specified major arc determines the reflex sector.