

Slices Build Volume

Explain volume by matching every cross-section.

HIGH SCHOOL • HSG-GMD.A.2

CCSS.Math.Content.HSG-GMD.A.2 “(+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.”

NAME _____

DATE _____

Materials and setup

Pencil. Use R for a sphere's radius and y for signed height above its center. The sphere and cylinder run from $y=-R$ to $y=R$. Two cones share their vertex at the center; their bases fill the top and bottom of the cylinder. Use n and exact expressions.

THE BIG IDEA

Cavalieri's principle compares solids with the same height and equal areas at every matching level. Compare a sphere with a cylinder of radius R and height $2R$ after removing the double cone. At a level y , only the upper cone ($y>0$) or lower cone ($y<0$) contributes a disk; at $y=0$ the cone slice is a point. Its radius is $|y|$. The remaining section is an annulus.

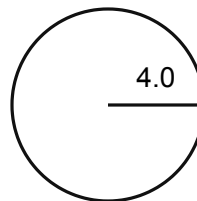
WORKED EXAMPLE

At $R=5$ and $y=3$, compare the two printed cross-sections. Why do their areas match?

1. The sphere slice radius is 4 since $4^2+3^2=5^2$. Its disk area is 16π .
2. The cylinder slice radius is 5; the cone slice radius is 3. The remaining annulus has area $25\pi-9\pi=16\pi$.
3. These equal areas illustrate one level. Cavalieri needs equality at every level, not just this example.

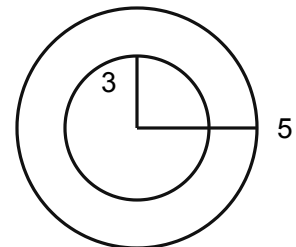
Answer: Both areas are 16π square units.

Sphere slice



Disk

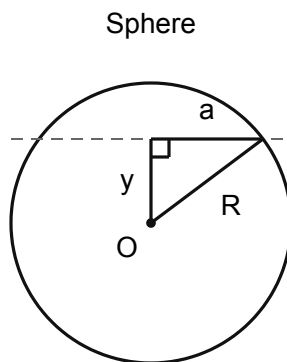
Cylinder minus cone



Region between circles

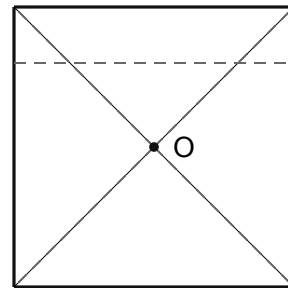
PRACTICE 4 problems

- 1** In the axial views, let a be the sphere slice radius at signed height y . Explain why $a^2 + y^2 = R^2$. Then derive the sphere slice area for every $|y| \leq R$.



Axial sections

Cylinder / double cone



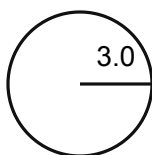
Common vertex at O

- 2** For the double cone, explain why its slice radius at height y is $|y|$. Find the area left in the cylinder slice and compare it with Practice 1.

- 3** Use the equal-slice argument to derive the sphere volume. A cone of base area B and height h has volume $Bh/3$. Show the cylinder volume and the total removed cone volume.

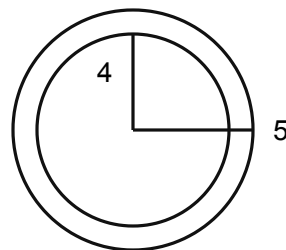
- 4** At $R=5$ and $y=-4$, use the printed sections to check their areas. Explain why you subtract one cone disk at this level, not two.

Sphere slice



Disk

Cylinder minus cone



Region between circles

APPLY IT Show your work

- 1** A right prism and an oblique prism stand between the same two horizontal planes, 8 cm apart. Every horizontal slice of each has area 12 cm^2 . Use Cavalieri to justify the oblique prism's volume; state why slant does not change it.

- 2** A cylinder and a sphere both have radius 6 cm and height 12 cm. They match at the center slice, but are their volumes equal? Compare the top slices, then use the sphere derivation.

CHALLENGE One step up

A solid has the same height h as a right circular cylinder of radius r . Every horizontal slice of the solid has area πr^2 , even though the slices have different shapes. Give an informal argument for its volume. Does equality of slice perimeters alone justify the same conclusion?

The double cone is two cones in opposite halves of the cylinder with a common vertex at the center, not two overlapping cones at each height. Require equality at every matching level and equal overall heights. Accept equivalent informal derivations that identify the sphere disk and cylinder-minus-cone annulus; the displayed axial views are vertical sections, not the horizontal slices themselves.

PRACTICE 1

A radius from O to the sphere boundary is the hypotenuse R of a right triangle with legs $|y|$ and a . Thus $a^2 + y^2 = R^2$, so the disk area is $\pi a^2 = \pi(R^2 - y^2)$. At $y = \pm R$ the area is 0.

PRACTICE 2

Each cone has height R and base radius R . Similar axial triangles give slice radius $|y|$. The annulus area is $\pi R^2 - \pi |y|^2 = \pi(R^2 - y^2)$, equal to the sphere slice at every height.

PRACTICE 3

By Cavalieri, sphere volume equals cylinder minus double cone.
 $V = \pi R^2(2R) - 2(\pi R^2 R/3) = 2\pi R^3 - (2/3)\pi R^3 = (4/3)\pi R^3$.

PRACTICE 4

Sphere disk area is $\pi(3^2) = 9\pi$. Cylinder minus cone gives $\pi(5^2) - \pi(4^2) = 9\pi$. At this negative height, only the lower cone is present; the upper cone lies at nonnegative heights.

APPLY IT 1

The oblique prism has volume 96 cm^3 .
 Both have the same height and the same 12 cm^2 slice area at every matching level. Cavalieri makes their volumes equal. The right prism is $Bh = 12 \times 8 = 96 \text{ cm}^3$; moving slices sideways does not change their areas.

APPLY IT 2

No. The sphere volume is $288\pi \text{ cm}^3$, while the cylinder volume is $432\pi \text{ cm}^3$.
 At the top $y=6$ the sphere slice has zero area; the cylinder slice has area 36π . Equality at one level is insufficient. From the derivation, $V_{\text{sphere}} = (4/3)\pi(6^3) = 288\pi$.

CHALLENGE

Cavalieri compares slice areas, not shapes: all matching areas equal πr^2 , so the solid volume is $\pi r^2 h$. Equal perimeters alone do not imply equal areas. For example, a square of side 1 and a 1.5-by-0.5 rectangle both have perimeter 4, but areas 1 and 0.75.